The Optimal Auditing Timing in the Repeated Principal Agent Model

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Abstract

The main result of this paper is that, in continuous time games with imperfect monitoring it is better to average information over time rather than respond at every instant. The two main reasons why it is better to introduce delayed response to signals are that it helps to (1) loosen promise-keeping and incentive compatibility constraints and (2) to increase the power of statistical test to detect deviation. In games with symmetric equilibrium, the equilibrium strategies are trigger strategy. The simplicity of the trigger equilibrium allows us to get computational results for this case.

1 Introduction

This paper studies timing issues in the repeated principal agent model. The principal agent model is used in corporate finance, labor, and macroeconomics. I want to address questions such as how often the compensation committee of the board of directors should meet to evaluate the CEO’s performance and how often a worker should be reviewed.

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Although most models of the principal agent relationship assume time is discrete, recent work suggests that working in continuous-time may have substantial advantages. These papers show that using continuous-time can actually simplify the analysis, allowing equilibrium sets to be characterized by tractable differential equations. But this literature assumes that observation takes place continuously and players have to react to this flow of information instantaneously. Is this indeed the optimal treatment of information? Suppose an assistant professor is hired. He works all the time but effort is hidden from the senior faculty. What the senior faculty observes is some noisy signal of the assistant professor’s performance. Should the senior faculty evaluate the junior faculty at every instant or not? What we observe in real life is that the performance of the assistant professor is evaluated at two-or-three-year intervals. Is this suboptimal or not?

A similar question of optimal treatment of endogenized information has been addressed in the discrete-time setting by Abreu, Milgrom, and Pearce (1991) and by Kandori (1992). In particular, Abreu, Milgrom, and Pearce (1991) (AMP) establish that, provided that the discount factor is large enough, it is beneficial to have reporting delays. Here I focus on the continuous-time setting with Brownian noise, which makes the games more tractable and the endogenized timing more natural.

What this paper establishes in the continuous-time setting is that committing to a lower frequency of observation can help to improve the outcomes of the games in the sense of expanding the set of outcomes achievable as an equilibrium. In other words, contracts where the manager evaluates the worker only once in a while, rather than continuously, have higher expected value for the principal even if the observation is costless. Unlike in AMP the continuous-time setting allows us to obtain solutions not just in some limiting case for high discount factors but for any value, at least in the case of games with grim-trigger strategy equilibrium, and point to ways of finding solutions in a more general setting.

There is a well-developed theory of repeated games with imperfect observation in discrete-time. Among others I would mention Abreu, Pearce, and Stacchetti (1990) (APS) and Fudenberg, Levine, and Maskin (1994). This literature develops quite general methodology for treating this class of games, characterizing limiting results for different values of the discount factors. I

show how some of the tools developed for the discrete-time games can be applied to analyzing continuous-time games.

In the first section, I informally present the main results of the paper and explain the methodology. In the second section, I show that, in a quite general principal agent model, when the principal can choose a frequency with which to observe the outcome of the agent’s performance, the profits that the principal achieves are higher. The benefits of delay here appear from the more efficient punishments on the boundary of the individually rational set.

In the last section, I concentrate on a reduced principal agent model where the equilibrium is a grim trigger strategy. In this case I show that committing to a lower frequency of observation is again beneficial, and I demonstrate a way to compute the expected values achievable in the best case. The reasons why the set gets expanded in the games with symmetric punishments are (1) the higher statistical power of the tests when the players wait significantly long enough, and (2) the uncertainty over the signal’s outcome and, thus, sluggishness in the response to the signal. In some sense these two effects follow from the AMP work extended to the games taking place in continuous time.

2 Overview

In this section I will present the main results of the paper and explain the logic behind them. Sannikov and Williams in a series of papers showed how to compute the equilibrium set for a quite general class of games played in continuous-time. Consider a stylized principal agent model played in continuous-time. Suppose that at each time $t \geq 0$ both the principal and the agent make the choice of some actions $a_{1t}$ and $a_{2t}$. The first player is the agent and the possible actions are to work hard or to shirk, $a_{1t} \in \{w, s\}$. The second player is the principal and the actions he can take are either to employ the agent or to fire him, $a_{2t} \in \{f, e\}$. If $(a_1, a_2)$ is a realization of the equilibrium strategies, then the realized payoff of the players is computed as

$$U_i(a_1, a_2) = r \int_0^\infty e^{-rt} q_i(a_{1t}, a_{2t}) dt,$$
where \( r \) is a common discount rate and \( q_i(a_{1t}, a_{2t}) \) takes value from the matrix

<table>
<thead>
<tr>
<th></th>
<th>fire</th>
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<tr>
<td>work hard</td>
<td>0, 0</td>
<td>( \pi, \pi )</td>
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<tr>
<td>shirk</td>
<td>0, 0</td>
<td>( \pi + g, -b )</td>
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with \( \pi, g, \) and \( b \) being positive. The fact that both the principal and the agent get \( \pi \) if they cooperate and zero otherwise is just a normalization. This is not a general version of the principal agent relationship but it is quite suitable for a brief exposition of the results. The payoffs are not observed directly but the action of the agent generates the signal

\[
X_t = \int_0^t f(a_{1t}) ds + \sigma B_t,
\]

where \( B_t \) is a Brownian motion and

\[
f(a_{1t}) = \begin{cases} 
\mu > 0 & \text{if } a_{1t} = w \\
0 & \text{if } a_{1t} = s 
\end{cases}
\]

The equilibrium outcomes of the repeated principal agent model played in continuous-time, just like any other repeated game, can be characterized by the set of expected values that they can achieve in the equilibrium. Let

\[
W_{it}(a_1, a_2) = E \left[ r \int_0^\infty e^{-rs} q_i(a_{1t}, a_{2t}) ds \bigg| \mathcal{F}_t \right]
\]

be the expected value that player \( i \) can get in an equilibrium and let \( W \equiv [W_1, W_2] \) be the vector of expected continuation payoffs of the players. Due to the normalization, these payoffs are bounded by the feasible set, the convex hull of the values from the payoff matrix. It is not a problem, given the usual assumptions about the principal, to support the values \((0, 0)\) and \((\pi + g, -b)\). Supporting \((\pi, \pi)\) takes more work as the agent has to have incentives not to deviate to shirking. Suppose that the signal \( X_t \) is observed in continuous time. Then we may use the results of Sannikov and Williams to compute the efficient frontier of the equilibrium set. In Figure 1 this set is marked with the letter \( C \).
To construct the frontier of this set we solve a second-order ordinary differential equation. Just like in APS to characterize the players’ actions we need to know only the expected continuation values. To support the values on the efficient frontier, we may concentrate on an equilibrium where the players move along the boundary of the set adjusting their continuation payoffs according to

\[ dW_t = r(W_t - q(a_{1t}, a_{2t}))dt + r\Psi(W_t)(dX_t - f(a_t)dt), \]

where \( W_t \) is the current promised value, \( q(a_{1t}, a_{2t}) \) is the instantaneous payoff that the players are supposed to obtain at time \( t \) on the equilibrium path, \( \Psi(W_t) \) is a vector which depends on the current promised value, and \( dX_t - f(a_t)dt = \sigma dB_{1t} \). Because the last term is a Brownian motion, the second part of the last equation is a martingale difference (i.e., has conditional expectation equal to zero).

The vector \( \Psi(W_t) \) provides incentives to the agent. To see how it works suppose the equilibrium implies that the agent is supposed to work at time \( t \), but the agent deviates to shirking. The benefit of such a deviation to the agent is \( rg \ dt \). The above expression \( dX_{1t} - f(a_{1t})dt \) becomes

\[ dX_{1t} - f(a_{1t})dt = (f(a'_{1t}) - f(a_{1t}))dt + \sigma dB_{1t} = -\mu dt + \sigma dB_{1t}, \]

which is no longer a driftless Brownian motion. Then if \( \Psi_1(W_t) = \frac{a}{\mu} \) the first player’s continuation payoff would be adjusted according to

\[ dW_t = r(W_t - q(a_{1t}, a_{2t}))dt + r\left(\frac{a}{\mu}\right)(dX_t - f(a_t)dt), \]

where \( \phi(a_{1t}, a_{2t}) \) is the instantaneous payoff that the players are supposed to obtain at time \( t \) on the equilibrium path, \( \Phi(W_t) \) is a vector which depends on the current promised value, and \( dX_t - f(a_t)dt = \sigma dB_{1t} \). Because the last term is a Brownian motion, the second part of the last equation is a martingale difference (i.e., has conditional expectation equal to zero).
\[ dW_{1t} = r(W_{1t} - q_1(a_1,t,a_2,t))dt + r\frac{g}{\mu}(-\mu dt + \sigma dB_{1t}), \]

instead of

\[ dW_{1t} = r(W_{1t} - q_1(a_1,t,a_2,t))dt + r\frac{g}{\mu}(\sigma dB_{1t}), \]

if there were no deviation. The first part and the third part of the equation are exactly as before, but the second part has a drift \(-rg dt\). So if the player chooses to deviate, the gain from the deviation \((rg dt)\) is offset by the drift in the observation process, lowering the future expected payoff, and the player is just indifferent between deviating or playing the equilibrium strategy. The other value of the vector \(\Psi(W_t)\) is designed to keep the payoffs on the frontier of the equilibrium set.

So far, we have characterized the equilibrium set \(C\) achievable when observation is continuous. Now suppose that the principal can choose the frequency with which the signal’s outcome is observed. Suppose that some set \(A\) of equilibrium payoffs is achieved in this case. What I want to demonstrate is that the set \(A\) can be strictly bigger than the set \(C\), \(C \subset A\). I will do so numerically. Concentrating on one particular contract, I will be able to compute the set \(B\) of values that can be supported in the equilibrium. Although the set \(B\) will not be the biggest set of equilibrium payoffs, \(C \subset B \subseteq A\). Which establishes that \(C \subset A\), i.e., that it is beneficial for the principal to observe the outcomes less frequently.

Unfortunately, we will not be able to compute the biggest set \(A\) in a general setting when the set of achievable payoffs is characterized with some frontier. Suppose that we restricted the players to participate only in the contracts where \((s,f)\) or \((w,e)\) are played on the equilibrium path. In this case we are able to characterize the biggest set \(A\). By concentrating on just the two actions, we reduce the set of payoffs achievable in the equilibrium only to the line segment between the points \((0,0)\) and \((\pi,\pi)\), and the optimal contract will essentially represent a grim-trigger strategy where, upon receiving a low enough signal, the players trigger the punishment of playing \((s,f)\) forever. Then the set \(C\) achievable when observation is continuous is just the origin \((0,0)\) (see Figure 2). If the principal chooses to observe with some frequency \(\Delta\), then some bigger set \(B\) would be achievable, and if \(\Delta\) is chosen optimally then we obtain the biggest equilibrium set, i.e., \(B = A\).
Consider the principal agent model in continuous-time. Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion on probability space $(\Omega, \mathcal{F}, P)$. The output evolves according to

$$dX_t = f(A_t)dt + \sigma dB_t,$$

where $A_t$ is the effort of the agent at time $t$, $\sigma$ is a constant, and

$$f(A_t) = \begin{cases} 
\mu > 0 & \text{if } A_t = w \\
0 & \text{if } A_t = s 
\end{cases},$$

where $\mu$ is the drift of the output process. The agent’s effort is the process $A = (A_t)_{t \geq 0}$ adapted to some filtration $\mathcal{F}_t$, where $A_t$ takes values in $\mathcal{A} = \{w, s\}$, an action space for the agent, to work or to shirk. Let $A$ be the random process of the agent’s effort and $a = (a_t)_{t \geq 0}$ denote a realization of the process.

The output $X_t$ is not observed all the time, but rather at the moment of the last observation $t$, the principal can choose the length of the observation window $\Delta_t$ and then the next observation will take place at the moment
If for some interval \( s \in [t_1, t_2) \), \( \Delta_s = 0 \), that means that the observation takes place continuously during this period. Once \( \Delta_t \) is chosen it cannot be changed, that is, we forbid the principal to change the decision after it was announced. Let \( D_t \) denote the last time \( \Delta_t \) was chosen. Let me illustrate how it works with an example. Suppose at time \( t \) the last observation took place and the principal sets some \( t \); then \( D_s = t \) during this interval. We assume that all the timing decisions are observed by the agent.

Then, given the pair of processes \( \Delta = (\Delta_t \geq 0)_{t \geq 0} \) and \( D = (D_t \geq 0)_{t \geq 0} \), let \( \mathcal{G}_t \) be the \( \sigma \)-algebra generated by the history of public observations of \( X_t \). Let there be a randomizing device generating a filtration \( \mathcal{F}_t \) and let \( \mathcal{F}_0 = \{\emptyset, \Omega\} \), a trivial \( \sigma \)-algebra. Then the available information by time \( t \) is summarized by the sigma algebra: \( \mathcal{F}_t = \mathcal{F}_0 \lor \mathcal{F}_t \lor \mathcal{G}_t \).

Given \( A \), a process of the agent’s effort on the filtered space \( \{\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}\} \), let \( P \) be the probability measure on this space. Let \( P \) be the probability measure implied by some alternative effort \( \tilde{A}_t \).

Before the agent starts working for the principal, the principal announces the contract specifying the processes \( \Delta \) and \( U \), where \( U = (U_t \geq 0)_{t \geq 0} \) denotes the process of the agent’s consumption.

The utility the agent gets is

\[
W_t = \mathbb{E} \left[ \int_t^\infty e^{-rs} [U_s - \alpha f(A_s)] ds \bigg| \mathcal{F}_t \right],
\]

where \( 0 < \alpha < 1 \) is a constant. Notice that, unlike in the other sections, we are not normalizing the utility here by multiplying it with \( r \). The principal values the following stream:

\[
\mathbb{E} \left[ \int_t^\infty e^{-rs} [dX_s - g(U_s) ds] \bigg| \mathcal{F}_t \right],
\]

where \( g(U_t) \) is the inverse of the utility function. Both the principal and the agent share a common discount rate \( r \). Let the reservation utility of the agent be \( v_0 \). We allow the principal to fire the agent. In this case the agent gets his reservation value and the principal gets zero profit.

**Definition 1** The optimal incentives-compatible contract is the triple of the processes \( D, \Delta \) and \( U \), which maximize the principal’s profit

\[
\mathbb{E} \left[ \int_0^\infty e^{-rt} [dX_t - g(U_t) dt] \right]
\]

8
subject to delivering the value $\hat{W}$ to the agent

$$E \left[ \int_0^{\infty} e^{-rt}[U_t - \alpha f(A_t)]dt \right] \geq \hat{W}$$

and subject to the incentives compatibility of effort $A$

$$E \left[ \int_t^{\infty} e^{-rs}[U_s - \alpha f(A_s)]ds \bigg| \mathcal{F}_t \right] \geq E_{P_{\tilde{A}_t}} \left[ \int_t^{\infty} e^{-rs}[U_s - \alpha f(\tilde{A}_s)]ds \bigg| \mathcal{F}_t \right]$$

for all deviations $\tilde{A}$ by the agent.

### 3.1 Continuous observation

To demonstrate the benefits of the lower frequency of observation let us consider a more specific setup, so that we are able to make numerical computations. Let the drift of the output process be $\mu = 1$ and define $g(u) = e^u - 1$, which is essentially the inverse of $u = \ln(c)$. First let us compute the optimal contract if the principal chooses to observe continuously, $\Delta_t = 0$, for all $t \geq 0$. Let us construct the set $C$ of the feasible payoffs that the players can get in the equilibrium. Essentially this set is characterized with an upper boundary $\pi(w)$, the principal’s profits if the agent’s promise value is $w$. The main result in Sannikov and Williams is that the efficient frontier in this case can be characterized by a second-order differential equation:

$$\pi''(w) = \min_{\{a,u\}} \frac{r\pi(w) - f(a) + g(u) - \pi'(w)(rw - u + \alpha f(a))}{(\alpha \sigma)^2}.$$  \quad (1)

This set is illustrated in Figure 3. On the horizontal axis is the agent’s expected lifetime utility and on the vertical axis is the corresponding principal’s lifetime profits/losses. The feasible set lies below the curve corresponding to the case where the observation is perfect. Below it is bounded by the case where the agent is not required to work and the principal just gives him some lifetime utility by providing a constant stream of consumption. Notice that the two curves cross at some point on the right. That means that if the principal chooses to give the agent a stream of utility to the right of that point, the best way to do so is just to give him a constant stream of utility
and not force him to work, which can be interpreted as a retirement. So, to compute the efficient frontier one should start at the point \((v_0, 0)\) and then obtain the frontier using (1), choosing the initial slope so that the frontier hits the retirement point.

\[
\text{Fig.3}
\]

To see what happens at the point of the reservation value \(v_0\), we shall use another result of Sannikov, which is that the law of motion of the agent’s promised value along the efficient frontier (if one wants the agent to exercise effort) is given by

\[
dW_t = (rW_t - u(W_t) + \alpha f(a))dt + \alpha (dX_t - f(a)dt).
\]  

(2)

As one can see, the part \(dX_t - A_t dt\) is equal to \(\sigma dZ_t\), which is a Brownian motion, and the first part is drift which, depends on the current promised value. Then, consider what happens on the boundary of the individually rational set, at the point \(v_0\). One of the properties of the Brownian motion is that, no matter how big the drift is, the Brownian motion can always drift the promised value to the left of the point \(v_0\) with positive probability. However, \(v_0\) is the reservation value, so the only way to deliver the promise \(v_0\) is to fire the agent, in which case the principal gets zero profits. The fact that we
can only obtain zero profits with continuous observation is quite important in this paper.

In order to compute the efficient frontier, one should start at the point \((v_0, 0)\) and then obtain the frontier using (1), choosing the initial slope so that the frontier hits the retirement point.

3.2 Flexible frequency of observation

Now suppose that the principal is free to choose any process \(\Delta_t\). Suppose that at time \(t\) the principal commits to observe the signal with a delay equal to \(\Delta_t\), then at time \(t + \Delta_t\), the players observe the signal \(y = x_{t+\Delta_t} - x_t\) (where \(x_t\) and \(x_{t+\Delta_t}\) are the realizations of the observation process \(X\) at times \(t\) and \(t + \Delta_t\)). As we are formulating the process recursively, \(y\) is the only relevant signal. Suppose that the principal wants to deliver the expected value \(v\) to the agent. Let \(\pi(v)\) be the function showing the maximum profit that the principal can get if the agent’s promised lifetime utility is \(v\). Let us concentrate on the equilibria where the agent is to exert fixed effort \(a_m = \omega\) for \(m \in [t, t + \Delta_t]\) or \(a_m = s\) for \(m \in [t, t + \Delta_t]\), that is, we forbid any contracts where the agent would work part of the period and shirk the other part of the period. Define the function

\[
\psi(a_s) = \int_{t}^{t+\Delta_t} f(a_s) dy,
\]

which is the cumulative drift of the realization of the observation process. Then conditional on \(a_s\), \(y \sim N(\psi(a_s), \Delta \sigma^2)\) and the \(\phi(y|\psi(a_s), \Delta_t)\) is the pdf of the signal \(y\) conditional on the total amount of the time that the player worked and the length of the observation window \(\Delta_t\) set by the principal. Let the function \(w(v, y)\) give the continuation values, given the promise \(v\) and the observed signal \(y\). Let \(a_s', s \in [t, t + \Delta_t]\) be some alternative effort that the player can take. Finally, to obtain a recursive formulation of the problem, let us drop the subscript \(t\) from \(\Delta_t\). From now on we will treat it as a choice variable depending on the current promised value to the agent.

The following proposition establishes the problem that we have to solve to obtain the optimal contract.
Proposition 1  The best contract with a flexible observation period, restricted to the equilibria where the agent is supposed to work for the entire period, is a solution to the following problem:

\[
\pi(v) = \max_{\{u,a,w(v,y),\Delta\}} \left\{ (f(a) - g(u)) \left(1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} \pi(w(v,y)) \phi(y|\Delta, \Delta) dy \right\}
\]

subject to:
promise keeping constraint

\[
(u - \alpha f(a)) \left(1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} w(v,y) \phi(y|\Delta, \Delta) dy \geq v
\]

incentive compatibility constraint

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v,y) \left( \phi(y|\Delta, \Delta) - \phi(y|\Delta - \tau, \Delta) \right) dy \geq \frac{\alpha}{r} (1 - e^{-r\tau}), \forall \tau \in [0, \Delta].
\]

Proof. The goal is to show that there is no profitable deviation. Suppose that the last observation took place at time \(t_0\). By a one shot deviation principle, it is sufficient to check that there is no deviation inside of the current reporting period \(t \in [t_0, t_0 + \Delta]\). Suppose that the agent chooses to deviate to some alternative plan \(a'_s\), such that \(\psi(a_s) - \psi(a'_s) = \tau\) (which is given our normalization \(\mu = 1\) is the time during which the agent deviated from the proposed action \(a = w\)) and \(a'_s = a_s\) for all \(s \in [t_0 + \Delta, \infty)\). Then consider a process defined as:

\[
a''_s = \begin{cases} shirk, & \text{if } s \in [t_0, t_0 + \tau) \\
work, & \text{if } s \in [t_0 + \tau, t_0 + \Delta] \end{cases}.
\]

Evidently, \(\psi(a_s) - \psi(a''_s) = \tau\) and \(\int_0^{\Delta} e^{-rs} f(a''_s) ds \leq \int_0^{\Delta} e^{-rs} f(a'_s) ds\), for all processes \(a'_s\) such that \(\psi(a_s) - \psi(a'_s) = \tau\). As \(\alpha f(A_s)\) comes with a negative sign into the agent’s utility, for the future discussion we shall concentrate on the deviations \(a'_s = a''_s\).

For any deviation \(a'_s\), the following inequality has to hold for the contract to be incentives compatible:

\[
\int_0^{\Delta} e^{-rs}(u - \alpha f(a)) ds + e^{-r\Delta} \int_{-\infty}^{\infty} w(v,y) \phi(y|\psi(a_s), \Delta) dy \geq
\]
Reminding ourselves that \( \psi(a_s) = \Delta \) and \( \psi(a'_s) = \Delta - \tau \), we obtain the incentives-compatibility constraint:

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) (\phi(y|\psi(a_s), \Delta) - \phi(y|\psi(a'_s), \Delta)) dy \geq \alpha \frac{1 - e^{-r\Delta}}{r} - \alpha \frac{(e^{-r\tau} - e^{-r\Delta})}{r},
\]

which is the inequality (5). Then it also rules out deviations starting at any point \( t \in [t_1, t_1 + \Delta], t_1 > t_0 \). This is implied by the same constraint as the punishment now gets closer in time:

\[
e^{-r(\Delta + t_0 - t_1)} \int_{-\infty}^{\infty} w(v, y) (\phi(y|\Delta, \Delta) - \phi(y|\Delta - \tau, \Delta)) dy > \alpha \frac{1 - e^{-r\tau}}{r}, \forall \tau \in [0, \Delta],
\]

Thus, we conclude that there is no profitable deviation inside of the current reporting period \( t \in [t_0, t_0 + \Delta] \).

Although Proposition 1 establishes the problem that we have to solve, it is very inconvenient to apply, as the incentives compatibility constraint (5) is quite complicated. The following conjecture will simplify this constraint.

**Conjecture 1** If

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\Delta, \Delta) \frac{(y - \Delta)}{\Delta \sigma^2} dy \geq \alpha
\]

and

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) (\phi(y|\Delta, \Delta) - \phi(y|0, \Delta)) dy \geq \alpha \frac{1 - e^{-r\Delta}}{r},
\]
hold then the following inequality also holds:

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \left( \phi(y|\Delta, \Delta) - \phi(y|\Delta - \tau, \Delta) \right) dy \geq \frac{\alpha}{r} \left(1 - e^{-rt} \right), \forall \tau \in [0, \Delta].
\]

The conjecture turns out to be hard to prove, so we shall leave it just as a conjecture. However, for computations, I will assume that it holds. After the computations are done, it is a straightforward exercise to check that the constraint (5) holds for the whole range of \( \tau \in [0, \Delta] \).

Given the above conjecture, the problem becomes

\[
\pi(v) = \max_{\{u, a, w(v,y), \Delta\}} \left\{ (a - g(u)) \left(1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} \pi(w(v,y)) \phi(y|\Delta, \Delta) dy \right\}
\]

subject to:

promise keeping constraint:

\[
(u - \alpha a) \left(1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\Delta, \Delta) dy \geq v
\]

marginal incentive compatibility constraint:

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\Delta, \Delta) \frac{(y - \Delta)}{\Delta \sigma^2} dy \geq \alpha
\]

full incentives compatibility constraint:

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \left( \phi(y|\Delta, \Delta) - \phi(y|0, \Delta) \right) dy \geq \frac{\alpha}{r} \left(1 - e^{-r\Delta} \right)
\]

3.3 Computation

In this subsection I want to show that it is beneficial to have reporting delays in the game when \( v_0 > 0 \). The logic behind this computational exercise was discussed in the Overview section. We have already computed the set \( C \) where the observation is continuous. Suppose that there exists the biggest self-generated set \( A \), which in our case would be a solution of the problem (7)
subject to (8),(9),(10). Suppose that, by considering some particular delay strategy and parameterizing some of the choice variables (i.e., assuming some functional form for \( w(v, y) \)), one obtains a boundary of the set \( B \), which is bigger than the original set \( C \ (C \subset B) \). Then the obvious statement is that if \( C \subset B \) and \( B \subseteq A \), then \( C \subset A \). That is, having an option of choosing the frequency of the observation is strictly beneficial for the players in the principal agent model.

Consider a strategy of exercising a fixed delay on the boundary of the individually rational set, at the point \( v_0 \), and having no delay for all the other values \( v > v_0 \). That is, \( \Delta(v) \), the delay \( \Delta \) as the function of the promised value, is defined as

\[
\Delta(v) = \begin{cases} 
\Delta & \text{if } v = v_0 \\
0 & \text{if } v > v_0 
\end{cases}
\]

Suppose that \( \pi(v_0) \) is the maximum profit that the principal can obtain with the promise \( v_0 \). Then it is easy to construct this set for all the other values of \( v \) using the method of Sannikov and Williams by starting at the point \( (v_0, \pi(v_0)) \) and aiming at the retirement point.

Consider the following problem:

\[
\Pi(v_0) = \max_{\{u, a, w(v_0, y), \Delta\}} \left\{ (a - g(u)) \left( 1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} \Pi(w(v_0, y)) \phi(y|\Delta, \Delta) dy \right\}
\]

s.t.

\[
(u - \alpha a) \left( 1 - e^{-r\Delta} \right) + e^{-r\Delta} \int_{-\infty}^{\infty} w(v_0, y) \phi(y|\Delta, \Delta) dy = v_0
\]

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v_0, y) \phi(y|\Delta, \Delta) \left( \frac{y - \Delta}{\Delta \sigma^2} \right) dy \geq \alpha
\]

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v_0, y) \left( \phi(y|\Delta, \Delta) - \phi(y|0, \Delta) \right) dy \geq \frac{\alpha}{r} (1 - e^{-r\Delta}).
\]

Then \( \Pi(\cdot) \) describes a higher boundary of some self-generated set of the game if \( \Pi(v_0) \) is obtained as a solution to the above problem and follows the solution of Sannikov and Williams for the continuous-time observation for \( v > v_0 \).
In the above operator, we are maximizing over the instantaneous utility $u$, the recommended action $a$, the reporting delay $\Delta$, and the transfer function $w(y)$. My goal is to obtain a numerical solution. Maximizing over the first three parameters numerically does not present a significant problem. Maximizing over a function $w(y)$, on the other hand, is quite difficult. The function $w(y)$ gives continuation values for the agent given the signal $y$ (see Figure 4). Evidently, this function should be non-decreasing and bounded from below by $v_0$. In the case of continuous observation one can think of $w(y) = \alpha y + \text{const}$, i.e., a line with the slope $\alpha$. When the observation window is very short, the variance of the signal goes to zero. Therefore, approximating $w(y)$ with a linear function can work, as the probability of going below $v_0$ is very low. However, when $\Delta$ is far from being small, a linear contract does not work because of the risk of giving a continuation value below $v_0$.

![Figure 4](image)

Although I do not know the exact form of the function $w(y)$, for practical purposes I will parameterize it with a sigmoid function:

$$w(y; \overline{w}, \underline{w}, \kappa, b) = \underline{w} + \frac{\overline{w} - \underline{w}}{1 + \exp(-\kappa(y - b))}.$$  

This function has the $s$-form depicted in Figure 4. The parameters $\overline{w}$ and $\underline{w}$ give the upper and the lower bounds for the values of function. The parameter $b$ gives the point around which the function is bending, and $\kappa$ determines how steep the transition is. For example, if $\kappa \to \infty$, this becomes
a step function with where \( w(y) = w \) if \( y < b \) and \( w(y) = \overline{w} \) if \( y > b \). This functional parameterization again shrinks the set of achievable equilibria.

The solution to this problem is illustrated in Figure 5. As one can see, the result that we get strictly outperforms the result of the continuous observation. Therefore, the conclusion is that it is beneficial to have an option of choosing the frequency of observation.

![Fig.5](image)

Note that Conjecture 1 was not proved, but we used it for computations. However, after the computation is done, it is quite easy to come back and check that it was satisfied for all \( \tau \in [0, \Delta] \).

As one can see, there is definitely a benefit of exercising delay on the boundary, as, by doing so, we strictly overperform the results that we obtain with continuous observation. Notice that I do not claim that I found the maximum set of achievable payoffs \( A \). However, even this set \( B \) is strictly bigger than the continuous observation one, \( C \), so the maximum set found with delay should be bigger than the one in the purely continuous observation case.

Let us discuss the interpretation of the result. In the case of continuous observation, the only way of supporting the value \( v_0 \) was to fire the agent, in which case he gets his reservation utility and the principal gets zero profits.
The frontier of the set $B$ can be physically supported by a contract where the observation is continuous if the promised value is greater than the reservation value $v_0$. If the continuation value is $v_0$, then instead of firing the agent, as was done in the case of continuous observation, the principal introduces an observation window of length $\Delta$ and, at the end of this time interval, gives continuation values on the upper frontier of the set $B$. Intuitively we may think of it as a punishment or a probation period, where the agent keeps on working, receiving some low compensation, and at the end of the period is reintroduced to the normal work. The benefit for the principal of having a delay comes from the fact that the punishment of delaying the observation and making the agent work for a while with a low compensation is more efficient than firing the agent.

If the length of the observation window $\Delta$ on the boundary $v_0$ is shrunk to a value much lower than the optimal, then the expected profit that the principal is getting also shrinks. To intuitively understand why it happens, consider the properties of signal $y$ when $\Delta$ is low. Remember that the conditional on working, distribution of the signal is $y \sim N(\Delta, \Delta \sigma^2)$. Suppose that the agent decides to shirk for the entire period. Then the distributions is $y \sim N(0, \Delta \sigma^2)$. The continuation value $w(y)$ works in some sense similarly to a statistical test, punishing for the low values of the signal and rewarding for the high. The power of the statistical test is determined by the differences in the means and the standard deviations. The difference in the means of the signal is $\Delta$, and the standard deviation is $\sigma \sqrt{\Delta}$. For small $\Delta$ the power of the test drops significantly as the standard deviation grows at a higher order than the mean. If we are away from the boundary and $\Delta$ is small, it is never a problem to provide incentives just with a linear contract, $w(y) = \alpha y + \text{const.}$, where the reward and punishments are in some sense symmetric. However, we run into a problem when we want to support $v_0$. Since $w(y)$ is bounded from below by $v_0$, the only way to provide incentives is to give rewards for the high realizations of $y$. As the signal becomes more and more noisy when we shrink $\Delta$, we have to give higher promises for the agent with higher probability, but the high promises to the agent are associated with lower promises for the principal and, therefore, as $\Delta$ shrinks, the expected profits for the principal also shrink. If we further shrink $\Delta$, we run into a problem of satisfying the promise-keeping constraint, because the continuation values should lie above $v_0$ and the variance is of a higher order than the mean.

Another result is that, if $v_0 = 0$, then the delay does not help on the boundary. The reason for this can be seen by supposing that the actions
were discrete, that is, the agent could not change the action inside of the reporting period. The outcome of such a game is at least as great as the outcome of the game where the action is taken continuously, however even in this game, if \( v_0 = 0 \), only zero profits are achievable at this point. The following proposition formalizes this statement.

**Proposition 2 (worst punishments)** If \( u \geq 0 \) and \( v_0 = 0 \) then only zero profit is achievable when the agent is promised \( v_0 \).

**Proof.** On the boundary of the individually rational set the promise keeping constraint is

\[
(u - \alpha) \frac{(1 - e^{-r\Delta})}{r} + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\psi(a_s))dy = 0.
\]

Reorganize this into

\[
u \frac{(1 - e^{-r\Delta})}{r} + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\psi(a_s))dy = \alpha \frac{(1 - e^{-r\Delta})}{r} \]

and plug it into the full incentives compatibility constraint:

\[
e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) (\phi(y|\Delta) - \phi(y|0)) dy \geq u \frac{(1 - e^{-r\Delta})}{r} + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|\Delta)dy
\]

\[
0 \geq u \frac{(1 - e^{-r\Delta})}{r} + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi(y|0)dy
\]

\[
0 \geq u \frac{(1 - e^{-r\Delta})}{r} + e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi\left(\frac{y}{\sigma \sqrt{\Delta}}\right) dy.
\]

Because \( u \geq 0 \) and \( e^{-r\Delta} \int_{-\infty}^{\infty} w(v, y) \phi\left(\frac{y}{\sigma \sqrt{\Delta}}\right) dy \geq 0 \), the above inequality holds only for the static Nash equilibrium of the stage game. ■
4 Reduced principal agent model

To further characterize the model, let us introduce some restrictions. The agent’s possible actions are, as before, to work hard or shirk, \(a_{1t} \in \{w, s\}\). The second player is the principal and the actions he can take are either to employ the agent or to fire the agent, \(a_{2t} \in \{f, e\}\). If the principal employs then the agent gets a fixed salary \(U_t = \pi + g\); and he gets \(U_t = 0\) if the agent is fired and \(g(U_t) = U_t\). The cost of work for the agent is \(g\). The principal gets \(2\pi + g\) if the agent is employed and is working, and zero otherwise. This setup is equivalent to playing the following game:

\[
\begin{array}{c|cc}
\text{action} & \text{fire} & \text{employ} \\
\hline
\text{work hard} & -g, 0 & \pi, \pi \\
\text{shirk} & 0, 0 & \pi + g, -b \\
\end{array}
\]

Just as before the a signal is generated by

\[
X_t = \int_0^t f(a_{1t})ds + \sigma B_t,
\]

where \(B_t\) is a Brownian motion and

\[
f(a_{1t}) = \begin{cases} 
\mu > 0 & \text{if } a_{1t} = w \\
0 & \text{if } a_{1t} = s 
\end{cases}
\]

Again, the principal is choosing \(\Delta\) and \(A_2\). If \((a_1, a_2)\) is a realization of the equilibrium strategies, then the realized payoff of the players is computed as

\[
U_i(a_1, a_2) = r \int_0^\infty e^{-rt} q_i(a_{1t}, a_{2t}) dt
\]

where \(r\) is a common discount rate and \(q_i(a_{1,t}, a_{2,t})\) takes its value from the matrix above. Notice that we now have \(r\) multiplying the integral, this is done for normalization. The players maximize their expected payoff

\[
W_{i,t}(A_1, A_2) = E \left[ r \int_0^\infty e^{-rs} q(A_{1t}, A_{2t}) ds \bigg| F_t \right].
\]
The rest of the setup, including the definition of the optimal contract, is as before. I will concentrate on finding the equilibria where only \((w, e)\) or \((s, f)\) are played on the equilibrium path. That is, we forbid the principal and agent to participate in any contract where \((s, e)\) is allowed on the equilibrium path. This assumption is made so that the analysis is more tractable.

Because of this assumption and the normalization that both players get \(\pi\) if \((w, e)\) is played and 0 if \((s, f)\) is played, all of the expected payoffs are symmetric. Because \((s, f)\) is the Nash equilibrium of the stage game we can always support the equilibrium where \((s, f)\) is played forever. However, if we want to support \((w, e)\) being played in the dynamic equilibrium, we have to provide incentives for the agent not to cheat on the equilibrium path.

Concentrating on the equilibria where only \((w, e)\) or \((s, f)\) are played implies that the optimal contract is a grim-trigger strategy. Without a loss of generality we can assume that the principal does not change \(\Delta\) throughout the game (see Figure 6).

\[ \text{Fig. 6} \]

\[ 0 \quad 1\Delta \quad 2\Delta \quad 3\Delta \quad t \]

\[ \text{4.1 Discretized action} \]

To learn what happens to the above game when the signal is observed continuously \((\Delta \to 0)\), it is useful to study a completely discrete version of the game. Assume that, at time \(t = n\Delta\), as soon as the realization becomes known, the players choose some action (from the support of the action space) and cannot change it until time \(t = (n + 1)\Delta\). Let \(v\) be the highest possible value achievable in the equilibrium. As the Nash equilibrium of the stage game is also a minmax, the lower bound on the expected utility is \(\underline{v} = 0\). To make the problem recursive, notice, that in the presence of the randomizing device we may concentrate only on the extremal values of the set achievable in the equilibrium. Suppose at time \(t = n\Delta\) (an instant before the arrival of the new information) we want to support the value \(v\). Then we may ignore the information that we had before and only concentrate on the new
information. What matters is \( y = X_n - X_{(n-1)\Delta} \), a normally distributed random variable with constant variance and mean depending on the actions of the players. That is, if \((w, e)\) is played then, \( y \sim N(\mu \Delta, \sigma^2 \Delta) \); otherwise if \((w, f), (s, f), \) or \((s, e)\), is played, \( y \sim N(0, \sigma^2 \Delta) \). Then from Theorem 7 in APS, it follows that it is inefficient to give interior punishments. Therefore, to compute the best equilibrium, we use a tail test with \( y_0 \) being the cut-off value. If the realization of \( y \) falls below it we go to the punishment state; otherwise we cooperate.

The value of the best equilibrium is obtained as a solution to the following problem:

\[
v = \max_{\{y_0\}} \left\{ \pi (1 - e^{-r\Delta}) + e^{-r\Delta} v \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \right\}
\]

(11)

subject to

\[
(\pi + g)(1 - e^{-r\Delta}) + e^{-r\Delta} v \left( 1 - \Phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) \right) \leq v
\]

(12)

where \( \Phi(\cdot) \) is the cumulative distribution function of the standard normal distribution. We do not have to compute the payoff separately for the two players due to the normalization that both of them get \( \pi \) if they cooperate and 0 otherwise.

We now treat \( \Delta \) as an endogenous parameter and study what happens to the equilibrium set when we vary this parameter. In Figure 7 the relationship
The main idea of this subsection is to show that, when the signal is observed continuously, the possibility of cooperation disappears. For small \( \Delta \)'s the signal is too noisy, so the power of the test is too low to support cooperation. Just as in the previous section the standard deviation is \( \sigma \sqrt{\Delta} \) and the difference in the means of the signal when the agent is *shirking* and when the agent is *working* is \( \mu \Delta \). Then if \( \Delta \) is small the power of the test drops significantly and the punishments happen too often on the equilibrium path which destroys the cooperative equilibrium. For large values of \( \Delta \) the incentive to deviate is too big. In between there may be some region for which one can support cooperation. The following proposition formalizes this statement.

**Proposition 3** There exist positive \( \underline{\Delta} \) and \( \overline{\Delta} \) such that, if \( \Delta \in (0, \underline{\Delta}] \cup [\overline{\Delta}, \infty) \), then the cooperative equilibrium \( (w, e) \) cannot be supported.

**Proof.** We first rewrite the incentives compatibility constraint in a more suitable form. From equations (1) and (2) follows

\[
g \frac{(1 - e^{-r\Delta})}{e^{-r\Delta}} \leq v \left( \Phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right)
\]

or, equivalently,
Define the function
\[ k(y_0, \Delta) = -\frac{g}{\pi \cdot e^{-r\Delta}} + \frac{\left( \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}. \]

In order to satisfy the incentives constraint, it should be true that \( k(y_0, \Delta) \geq 0 \). The proof strategy is to show that, for the limiting values of \( \Delta \), it does not hold. To establish the proposition we will prove 3 lemmas.

**Lemma 1** \( y_0^* = \arg \max_{\{y_0\}} \left\{ \lim_{\Delta \to 0} k(y_0, \Delta) \right\} < 0 \)

**Proof.** Maximizing \( k(y_0, \Delta) \) with respect to \( y_0 \) is equivalent to maximizing just the second part of the expression. For \( \Delta \) small enough the following is true:
\[ \frac{\left( \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)} \approx \frac{\mu}{\sigma \phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right)} \sqrt{\Delta} + O(\Delta)^2. \]

Let us maximize the expression
\[ \frac{\mu}{\sigma \phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right)} \sqrt{\Delta} \]
\[ \frac{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}. \]

The first order condition with respect to \( y_0 \) is:
\[ \frac{\mu}{\sigma \sqrt{\Delta}} \frac{1}{\sigma \sqrt{\Delta}} \frac{\phi'\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right)}{\Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right)} \left(1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)\right) - e^{-r\Delta} \phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) \phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) \frac{\left(\Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)} \leq 0 \]

\[ \text{Since } \Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) = \int_{\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}}^{\frac{y_0}{\sigma\sqrt{\Delta}}} \phi(x) dx \]

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or, equivalently,

\[
\phi' \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) \left( 1 - e^{-r \Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \right) - e^{-r \Delta} \phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) = 0
\]

Because

\[
\phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right)^2}
\]

it follows that

\[
\phi' \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) = \left( - \frac{y_0}{\sigma \sqrt{\Delta}} \right) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right)^2} = \left( - \frac{y_0}{\sigma \sqrt{\Delta}} \right) \phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right)
\]

Plugging this into (13) we obtain

\[
\left( - \frac{y_0}{\sigma \sqrt{\Delta}} \right) \phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) \left( 1 - e^{-r \Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \right) - e^{-r \Delta} \phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) = 0
\]

which is equivalent to

\[
\left( - \frac{y_0}{\sigma \sqrt{\Delta}} \right) \left( 1 - e^{-r \Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \right) = e^{-r \Delta} \phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right).
\]

Therefore,

\[
\left( - \frac{y_0}{\sigma \sqrt{\Delta}} \right) = \frac{e^{-r \Delta} \phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right)}{\left( 1 - e^{-r \Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \right)} > 0
\]

This proves lemma 1. ■

**Lemma 2** Because \( \lim_{\Delta \to 0} k(y_0, \Delta) < 0 \), we cannot support a cooperative equilibrium if \( \Delta \) is too small.
Proof. Consider the following expression:

\[
\frac{\Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)}.
\]

The following is true:

\[
\Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right) \leq \frac{1}{\sqrt{2\pi} \sigma \sqrt{\Delta}}
\]

When \(\Delta\) is small enough by Lemma 1 guarantees that:

\[
1 - \frac{1}{2} e^{-r\Delta} \leq 1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)
\]

and so

\[
0 \leq \frac{\Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)} \leq \frac{1}{\sqrt{2\pi} \sigma \sqrt{\Delta}}
\]

Then by definition:

\[
\lim_{\Delta \to 0} k(y_0, \Delta) = \lim_{\Delta \to 0} \left[ -\frac{g}{\pi \cdot e^{-r\Delta}} + \frac{\Phi\left(\frac{y_0}{\sigma\sqrt{\Delta}}\right) - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)}{1 - e^{-r\Delta} \left(1 - \Phi\left(\frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}}\right)\right)} \right] \leq 0
\]

\[
\lim_{\Delta \to 0} \left( -\frac{g}{\pi \cdot e^{-r\Delta}} + \frac{1}{\sqrt{2\pi} \sigma \sqrt{\Delta}} \frac{\mu \sqrt{\Delta}}{1 - \frac{1}{2} e^{-r\Delta}} \right) = -\frac{g}{\pi} < 0
\]

which implies \(k(y_0, \Delta) < 0\)

This concludes the proof of Lemma 2. ■

Lemma 3 Because \(\lim_{\Delta \to \infty} k(y_0, \Delta) < 0\), we cannot support a cooperative equilibrium if \(\Delta\) is too large.

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Proof. Consider
\[ k(y_0, \Delta) = -\frac{g}{\pi \cdot e^{-r\Delta}} + \frac{\Phi \left( \frac{y_0}{\sqrt{\sigma}} \right) - \Phi \left( \frac{y_0 - \mu \Delta}{\sqrt{\sigma}} \right)}{1 - e^{-r\Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sqrt{\sigma}} \right) \right)}. \]

Because
\[ \lim_{\Delta \to \infty} \left( -\frac{g}{\pi \cdot e^{-r\Delta}} \right) = -\infty \]
and
\[ 0 \leq \lim_{\Delta \to \infty} \frac{\Phi \left( \frac{y_0}{\sqrt{\sigma}} \right) - \Phi \left( \frac{y_0 - \mu \Delta}{\sqrt{\sigma}} \right)}{1 - e^{-r\Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sqrt{\sigma}} \right) \right)} \leq 1, \quad \forall y_0 \]

it follows that
\[ \lim_{\Delta \to \infty} k(y_0, \Delta) = -\infty \]

This concludes the proof of Lemma 3 and Proposition 3.

4.2 Continuous-time action

We now drop the assumption that the players cannot change their actions within the periods \([n\Delta, (n + 1)\Delta]\). The players can change their actions at every instant but observe the realization of the signal with periodicity \(\Delta\) (now we may call \(\Delta\) a reporting delay).

Let us construct an equilibrium where the players cooperate all the time within each period of length \(\Delta\) if they cooperate at all. Call this type of equilibrium a constantly cooperative equilibrium. In contrast, there exist equilibria where the players cooperate during only part of the period. The next proposition establishes the problem that we have to solve to construct a constantly cooperative equilibrium. The second part of the proposition establishes that, if \(\Delta\) is a choice variable and is chosen optimally, then the equilibrium where the players cooperate only some of the time within the period is inferior to the one where they cooperate throughout the period.

Proposition 4 In the game with continuous action and reporting delays of length \(\Delta\) the best contract within the class of constantly cooperative equilibria is a solution to
\[
\max_{\{y_0\}} v = \pi(1 - e^{-r\Delta}) + e^{-r\Delta}v \left(1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right) \tag{14}
\]

such that, for \( \tau \in [0, \Delta] \),

\[
g(1 - e^{-r\tau}) + \pi(1 - e^{-r\Delta}) + e^{-r\Delta}v \left(1 - \Phi \left( \frac{y_0 - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) \right) \leq v \tag{15}
\]

Furthermore, if \( \Delta \) is a choice variable, for any equilibrium where the players cooperated only some part of the period of length \( \Delta' \), delivering the value \( v' \), one can always switch to a constantly cooperative equilibrium with a lower \( \Delta'' < \Delta' \), delivering the value \( v'' > v' \).

**Proof.** First, fix \( \Delta > 0 \) and suppose that \( y_0 \) solves the problem (14) s.t. (15). Let us show that, if the incentive compatibility constraint (5) is satisfied for all \( \tau \in [0, \Delta] \), then the agent does not have incentives to deviate. The recommended strategy \( A_{1t} \) for the agent has the following form:

\[
A_{1t} = w, \text{ during the first observation period (i.e., } t \in [0, \Delta])
\]

\[
A_{1t} = \begin{cases} 
  w & \text{if } y \geq y_0 \forall t \in [n\Delta, (n+1)\Delta] \text{ if } w \text{ was played in the previous period} \\
  s & \text{if } y < y_0, \forall t \in [\Delta, \infty)
\end{cases}
\]

The signal \( y \) which the agents get at time \( \Delta \) is normally distributed: \( y \sim N(\mu \Delta, \sigma^2 \Delta) \).

Suppose that, at time \( t = 0 \) the worker considers some deviation \( A_{1,t}' \) from the recommended strategy \( A_{1t} \). Let us show that the strategy

\[
A''_{1t} = \begin{cases} 
  A_{1t}' & \text{if } t \in [0, \Delta) \\
  A_{1t} & \text{if } t \in [\Delta, \infty)
\end{cases}
\]

does not improve the agent’s expected utility. Let \( \tau = \Delta - \frac{1}{\mu} \int_0^\Delta f(a''_t)dt \) be the measure of time during which the agent deviated during the first period. Alternatively, consider a deviation of the form

\[
A'''_{1t} = \begin{cases} 
  s & \text{if } t \in [0, \tau) \\
  w & \text{if } t \in [\tau, \Delta) \\
  A_{1,t} & \text{if } t \in [\Delta, \infty)
\end{cases}
\]
Then
\[ W_{1,0}(A''_{1,t}, A_{2,t}) = g(1 - e^{-r\tau}) + \pi(1 - e^{-r\Delta}) + e^{-r\Delta}v \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) \right) \geq W_{1,0}(A''_{1,t}, A_{2,t}). \]

The above inequality is true because the agent is always at least as well off if the deviation is done earlier (because of discounting) and the part \( e^{-r\Delta}v \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) \right) \) is the same for both strategies. But the constraint (15) implies that
\[ W_{1,0}(A''_{1}, A_{2}) \leq W_{1,0}(A_{1}, A_{2}) = v. \]

So there exists no alternative strategy \( A''_{1,t} \), which is some deviation during the period \( t \in [0, \Delta) \) and then following the recommended action forever, improving the agent’s expected utility at time zero.

Suppose that up to time \( t_0 \in [0, \Delta) \) the agent followed the recommended action and then considered to deviating to some strategy for the remainder of the period \([t_0, \Delta)\) and then following the recommended action. These deviations are ruled out in the same way, because the punishment gets closer in time. So we conclude that there is no "one-shot" deviation – a strategy which would improve the agent’s utility for any part of the period \([0, \Delta)\). Then by exponential discounting and the "one-shot" deviation principle we rule out any infinite deviation \( A''_{1,t} \). That proves that the contract is incentives compatible.

Next we have to prove the second statement, that switching to a constantly cooperative equilibrium is beneficial. By Theorem 7 in APS we know that we may concentrate on the extreme continuation values, and the best test to detect the deviation is a tail test. Suppose that there is an equilibrium determined by \( \{\Delta', y_0'\} \) delivering utility \( v' \) where the agent works for a measure of time \( \delta > 0 \). Let us show that there is an alternative strategy delivering a higher utility. The solution \( \{\Delta', y_0'\} \) should satisfy
\[ v' = \pi(e^{-r\delta} - e^{-r\Delta}) + e^{-r\Delta}v' \left( 1 - \Phi \left( \frac{y_0' - \mu (\Delta' - \delta)}{\sigma \sqrt{\Delta'}} \right) \right) \]
and the following inequality for all \( \tau \in [0, \Delta' - \delta] \)
\[ g(1 - e^{-r\tau}) \leq e^{-r\Delta' - \delta}v' \left( \Phi \left( \frac{y_0' - \mu(\Delta' - \delta) + \mu \tau}{\sigma \sqrt{\Delta'}} \right) - \Phi \left( \frac{y_0' - \mu(\Delta' - \delta)}{\sigma \sqrt{\Delta'}} \right) \right). \]
To see that those equations are to be satisfied, notice that, given a certain threshold $y_0$, it is most tempting to deviate in the beginning, as the punishment is too distant. So, the only possible equilibria where the agent is to work $\Delta' - \delta$ measure of time, during the period of length $\Delta'$, is where they play $(s, f)$ for $t \in [0, \delta)$ and $(w, e)$ for $t \in [\delta, \Delta')$.

Then consider shrinking the reporting delay from $\Delta'$ to $\Delta'' = \Delta' - \delta$, leaving the same test $y_0$. The incentive constraints should still bind (as the power of the tail test increase)

$$g(1 - e^{-r\tau}) \leq e^{-r\Delta' - \delta} v' \left( \Phi \left( \frac{y_0' - \mu(\Delta' - \delta) + \mu\tau}{\sigma\sqrt{\Delta' - \delta}} \right) - \Phi \left( \frac{y_0' - \mu(\Delta' - \delta)}{\sigma\sqrt{\Delta' - \delta}} \right) \right)$$

$$\forall \tau \in [0, \Delta' - \delta]$$

and then it is easy to see that

$$v' < v'' = \pi (1 - e^{-r\Delta' - \delta}) + e^{-r\Delta' - \delta} v' \left( 1 - \Phi \left( \frac{y_0 - \mu(\Delta' - \delta)}{\sigma\sqrt{\Delta' - \delta}} \right) \right).$$

Therefore, the value that we get from solving (14) s.t. (15) with delay $\Delta''$ is higher than $v'$. □

Proposition 4 gives us the condition that we have to satisfy in order to support cooperation and find the best equilibrium. Let me provide some intuition behind the formal proof. Concentrating on constantly cooperative equilibria means that the players take actions $(w, e)$ for a period of length $\Delta$. If the realization of the signal $y$ falls below $y_0$ the punishment of playing $(s, f)$ is triggered, otherwise they continue playing $(w, e)$. It is useful to consider some deviation inside of the first reporting period and following the proposed strategy for the rest of the game. As the agent is choosing the action in continuous-time, at any instant he can switch to any alternative strategy. Suppose that up to time $t \in [0, \Delta)$ he followed the equilibrium strategy and then for the rest of the period $(t, \Delta)$ switched to some alternative strategy. Let us first rule out all possible deviations starting from time $t = 0$. Suppose there exists some profitable deviation of this sort where the agent deviates for length of time $\tau \in [0, \Delta]$ during this period. Then notice that the time during which the agent deviated influences only the mean of the signal, and all deviations where the agent deviated for time $\tau$ produce signals with the same distribution. Because of discounting, it is always beneficial to have
the deviation earlier on rather than later. But all such deviations are ruled out by the incentives-constraints (15) holding for the whole range \( \tau \in [0, \Delta] \).

So, no deviation starting at time \( t = 0 \) is profitable. Then it also rules out all deviation where the agent wants to deviate from time \( t \in (0, \Delta) \) as the punishments get closer in time. Finally, by the single-deviation principle there exists no profitable deviation beyond the the first reporting period.

Next, I want to show that concentrating on the constantly cooperative equilibria is justifiable when the \( \Delta \) is chosen to maximize the expected payoffs. Notice that if one is considering an equilibrium where the players play \((s, f)\) during the measure of time \( \delta \) where \( 0 < \delta < \Delta' \) and \((c, e)\) the rest of the period during the measure of time \( \Delta' - \delta \), then again the best the agent can do is to play \( s \) for the first amount of time \( \delta \) regardless of what the equilibrium is. So, for it to be an equilibrium, the players are supposed to play \((s, f)\) for \( t \in [0, \delta) \) and \((w, e)\) for \( t \in [\delta, \Delta') \). Then compare this equilibrium to the constantly cooperative one with the time period being \( \Delta'' = \Delta' - \delta \). This alternative equilibrium is a much better one as (1) the players get the benefits of cooperation earlier rather then later, (2) the punishments come earlier rather then later which allows slacker incentives compatibility constraints, and (3) the test is of higher power in the case of the constantly cooperative equilibrium as in both cases under no deviations the signal \( y \) has the same mean \( \mu(\Delta' - \delta) \) but the variance under the constantly cooperative equilibrium is lower, i.e., \( \sigma^2(\Delta' - \delta) \) versus \( \sigma^2\Delta' \) in the case that we want to rule out, which makes all the deviations easier to detect and therefore a looser incentives-compatibility constraint is required.

Although Proposition 4 formulates the problem that we have to solve, it is not particularly helpful in terms of computations, as the incentives-compatibility constraint has to be satisfied for the whole range \( \tau \in [0, \Delta] \).

Proposition 5 shows that the problem can be significantly simplified and that it is necessary and sufficient to check the incentives only on the ends of the interval \([0, \Delta]\).

**Proposition 5** *The problem (14) s.t. (15) is equivalent to the problem:*

\[
\max_{\{y_0\}} v = \pi(1 - e^{-r\Delta}) + e^{-r\Delta} v \left( 1 - \Phi \left( \frac{y_0 - \mu\Delta}{\sigma\sqrt{\Delta}} \right) \right) \tag{16}
\]

\[s.t.\]
\[
gr - e^{-r\Delta} v \cdot \phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \frac{\mu}{\sigma \sqrt{\Delta}} \leq 0 \quad (\tau \to 0)
\]

(\pi + g)(1 - e^{-r\Delta}) + e^{-r\Delta} v \left( 1 - \Phi \left( \frac{y_0}{\sigma \sqrt{\Delta}} \right) \right) \leq v \quad (\tau = \Delta).

**Proof.** Let \( v^* \) be the maximum utility obtained as a solution to the problem (14) s.t. (15) and \( y_0^* \) be the maximizing parameter. Then rewrite the inequality (15):

\[
\frac{g}{v^*} \frac{(1 - e^{-r\tau})}{e^{-r\Delta}} \leq \left( \Phi \left( \frac{y_0^* - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) - \Phi \left( \frac{y_0^* - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right), \quad \forall \tau \in [0, \Delta].
\]

**Lemma 4** If \( y_0^* \) solves the problem (14) s.t. (15) then \( \frac{y_0^* - \mu \Delta}{\sigma \sqrt{\Delta}} < 0 \)

**Proof.** problem (14) s.t. (15) is equivalent to:

\[
\min_{\{y_0\}} y_0 \text{ s.t } v = \pi(1 - e^{-r\Delta}) + e^{-r\Delta} v \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right)
\]

\[
g \frac{(1 - e^{-r\tau})}{e^{-r\Delta}} \leq \left( \Phi \left( \frac{y_0 - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right), \quad \forall \tau \in [0, \Delta].
\]

Fix \( \tau \) and let \( v^* \) solve the above problem. Then \( \frac{g}{v^*} \frac{(1 - e^{-r\tau})}{e^{-r\Delta}} \) is just a constant and the difference between \( \frac{y_0 - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \) and \( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \) is also a constant \( \frac{\mu \tau}{\sigma \sqrt{\Delta}} \). The right hand side of (19) is a function of the two normal cdf’s. And it is maximized at \( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} = -\frac{1}{2} \frac{\mu \tau}{\sigma \sqrt{\Delta}} \) (just by inspection of the shape of the normal cdf). Then one can always pick a \( y_0 \) satisfying \( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \leq -\frac{1}{2} \frac{\mu \tau}{\sigma \sqrt{\Delta}} \leq 0 \). Which concludes the proof of the lemma. ■

Let

\[
f(\tau) \equiv \frac{g}{v^*} \frac{(1 - e^{-r\tau})}{e^{-r\Delta}}
\]

and

\[
h(\tau) \equiv \left( \Phi \left( \frac{y_0^* - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) - \Phi \left( \frac{y_0^* - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right).
\]

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For the small $\tau$ in order $(19)$ to be satisfied, it is necessary and sufficient that $f'(\tau) \leq h'(\tau)$ should hold and this is established by $(17)$.

The function $f(\tau)$ is increasing concave and the function $h(\tau)$ is increasing, initially convex (by Lemma 4) and then can turn concave. Let us show that on the concave part they can cross only once. To do so, let us establish the regions where $f'(\tau) \leq h'(\tau)$ and $f'(\tau) \geq h'(\tau)$.

\[ f'(\tau) \equiv \frac{gr}{v^*} \exp[-r(\tau - \Delta)] \]

and

\[ h'(\tau) \equiv \phi \left( \frac{y_0^* - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right) = \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{y_0^* - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right)^2 \right]. \]

Then $f'(\tau) \leq h'(\tau)$ is equivalent to

\[ \frac{gr}{v^*} \exp[-r(\tau - \Delta)] \leq \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{y_0^* - \mu \Delta + \mu \tau}{\sigma \sqrt{\Delta}} \right)^2 \right]. \]

As a logarithm is a monotonic transformation we can take logs of both sides. Then let $a_1 = r$, $b_1 = -r \Delta$, $a_2 = \frac{\mu}{\sigma \sqrt{\Delta}}$, $b_2 = \frac{y_0^* - \mu \Delta}{\sigma \sqrt{\Delta}}$, and $c = \ln \left( \frac{1}{\sqrt{2\pi}} \frac{v^*}{gr} \right)$, and the above inequality is equivalent to

\[ -\frac{1}{2} (a_2 \tau + b_2)^2 + (a_1 \tau + b_1) + c \geq 0, \]

which holds if and only if

\[ -\frac{1}{2} a_2^2 \tau^2 + (a_1 - b_2 a_2) \tau + b_1 + c - \frac{1}{2} b_2^2 \geq 0. \]

The left hand side of the inequality is a downward-sloping parabola, so it has at most two crossing points with the x-axis. They correspond to the
points where \( f'(\tau) = h'(\tau) \) (see Figure.8)

By (17) and lemma 4 we know that the first crossing point \( \tau_1 \) should satisfy \( \tau_1 \leq 0 \) and is on the upward sloping part of \( h'(\tau) \). The second crossing point \( \tau_2 \) is on the downward sloping part of \( h'(\tau) \). Then for \( \tau = \tau_1 \) \( f'(\tau) \leq h'(\tau) \), for \( \tau \in (0,\tau_2) \), \( f'(\tau) < h'(\tau) \), for \( \tau = \tau_2 \), \( f'(\tau) < h'(\tau) \), and for \( \tau \in (\tau_2,\infty) \), \( f'(\tau) > h'(\tau) \). Then again by (17) and Lemma 4 there is at most one point where \( f(\tau) = h(\tau) \) on the region \( \tau \in (\tau_2,\infty) \).

**Corollary 1** If \( \Delta \to 0 \) or \( \Delta \to \infty \) cooperation cannot be supported.

**Proof.** This follows directly from Propositions 3 and 5.

Proposition 5 is quite helpful in computations. For the parameters \( \pi = 5, g = 1, \mu = 1; \sigma = 3 \), and \( r = 0.05 \), let us compute \( v \) as a function of \( \Delta \).
As one can observe in Figure 9, delay is very helpful in this game, allowing support of cooperation, which was impossible when the observation was continuous. However, there is a significant loss of efficiency compared to the case when the action was fixed during the period (left plot in Figure 10). And the Figure 11 shows Figure 9 and Figure 10 combined.

These figures illustrate that it is beneficial to have a lower frequency of observation in the games with grim-trigger equilibria.
Notice that Propositions 4 and 5 imply that the incentives are not distributed evenly inside of the period. To understand this, suppose that the marginal incentives-compatibility constraint (17) is binding and (18) is slack. That would imply that at time \( t = 0 \) the agent is indifferent between shirking or working for a very short interval of time. Proposition 5 then would imply that, for any longer than marginal deviation, the agent strictly prefers to work. At all times \( t > 0 \) the agent strictly prefers working even to a marginal deviation.

The fact that the incentives are spread unevenly within the period imply that there is a potential for improvement by employing some better monitoring instruments (more in the discussion section). In addition if the action were not binary but continuous as presented by Sannikov and Skrzypacz (2005) in the model of oligopolistic competition with flexible production, these two propositions would imply that the equilibrium with a constant level of production would be impossible. Again, the possible remedies for this will be discussed in the last section of the paper.

4.3 Comparative statics

Now with this computational procedure we can answer several interesting questions about the optimal auditing timing in the principal agent model.

Noise. As we discovered in the previous subsection it is optimal to have a reporting delay. But how does the optimal window of observation, \( \Delta \), change with the quality of the signal?

When the observation is perfect, that is, \( \mu > 0 \) and \( \sigma = 0 \), it is evident, that there exists some \( \overline{\Delta} \), so that, for all \( \Delta \in (0, \overline{\Delta}) \), we can achieve maximum cooperation, \( \nu = \pi \). If the observation is imperfect, that is \( \mu > 0 \), and \( \sigma > 0 \), what matters is the signal-to-noise ratio \( \frac{\mu}{\delta} \). Although, it is hard to do the analytic comparative statics, because the optimal \( \Delta \) shows up as a result of optimization problem, we can use the computational results to show that the time interval between the successive audits goes down. For instance, for the parameter values fixing \( \pi = 5, g = 1, \mu = 1 \), and \( r = 0.05 \) we can compute
the value of the optimal $\Delta$ as a function of noise $\sigma$ (see Figure 12).

![Figure 12](image)

This is an interesting result, because the reason for the delay in the first place was to get the higher power of the test. However, as the signal becomes more noisy the optimal $\Delta$ goes down.

**Patience.** What happens to the optimal observation period as the players become less patient? To answer this question we shall again use simulation results.

![Figure 13](image)

Figure 13 shows the relationship between the discount factor $r$ and the optimal observation period $\Delta$, for the parameter values $\pi = 5, g = 1, \mu = 1,$
and $\sigma = 1$. As one can see, as the players become more impatient the frequency of auditing goes up.

**Costly observation.** Throughout the paper we assumed that the observation has no cost. Then an interesting question to ask is: what happens when the observation is costly, which is a reasonable assumption in many principal agent applications. Suppose that we are in our reduced setting and every time the principal observes he pays a certain cost $c$. If the principal chooses a certain $\Delta$ the agent’s problem is unchanged. Suppose that after solving the agent’s problem the principal sets the cut-off value for the signal $y_0$ and the agent’s expected value is $v$. Having introduced the cost of observation the principal’s problem is no longer the same as the agent’s, since the principal bears additional cost. This additional cost for the principal can be computed using the following equation:

$$v_c = c \frac{e^{-r\Delta}}{1 - e^{-r\Delta} \left( 1 - \Phi \left( \frac{y_0 - \mu \Delta}{\sigma \sqrt{\Delta}} \right) \right)}.$$  

Then the principal’s expected value is $v_p = v - v_c$. Again, as in the previous subsection, we may concentrate on the constantly cooperative equilibrium, which follows from the same argument that we applied in Proposition (4), where we showed that we can always switch to a constantly cooperative equilibrium with a lower $\Delta$, delivering a higher expected payoff.
Figure 14 shows the principal’s expected values as a function of $\Delta$, when the cost parameter is equal to $c = 0, 0.1, 0.2$ (from the upper to the lower lines). Introduction of costly observation makes the continuous observation even more problematic and shifts the optimal $\Delta$ to the right. In other words as the cost to observe increases the optimal delay $\Delta$ goes up as well. However, even with cosless observation, it is beneficial to have a delay.

## 5 Conclusion and discussion

The results in the previous sections establish that having a periodic reevaluation of the partnership’s performance improves the set of payoffs achievable in equilibrium over a case of continuous observation. In a general case the I showed that a contract of higher value can be constructed, if a flexible reporting delay is introduced. In a more specific case when the equilibrium is a grim-trigger strategy, we are able to fully characterize the set of payoffs achievable in the equilibrium and get the computational results.

Discussing the results it is important to notice that we leave some questions unanswered. In particular, it is easy to see that the incentives to
work are not distributed evenly inside the period. Usually it is the marginal incentives-compatibility constraint that holds even for time zero and strictly binds forever after. So there should be a way to improve upon it changing the observation structure and allowing the agents to communicate. One way is to allow the principal to observe the signal privately and if the performance of the agent is unsatisfactory, to stop the play and skip to the punishment phase of the equilibrium. In my separate working project "The benefits of private monitoring and options to stop in a continuous-time principle agent model" I have started studying this case.

Sannikov and Skrzypacz (2005) consider an oligopolistic game, and one of their results is that, if the observation is continuous, no collusion is possible, which is analogous in some way to Proposition 3 in this paper. It seems likely that reporting delay may increase the set of achievable equilibria and allow some collusion in their setting. However, if the reporting delay is applied in the same way it is here, then the equilibrium would most likely have some unattractive features because of the uneven spread of incentives within the period (such as varying quantities of output within the period). It seems like some of these unattractive features could be remedied by conditioning not on the final realization of the signal process at the end of the period but on the whole path, or by having a delegated monitoring, where some committed stopping device observes the original signal and informs players which strategies to follow. That is analogous to the private monitoring and the option to stop in the principal agent model that I am currently studying.

It would be interesting to explore other types of contracts, such as the contracts with random stopping time.

6 Literature cited

4. Sannikov, Y. 2004 "Games with Imperfectly Observable Actions in continuous-time" mimeo

5. Sannikov, Y. 2003 "A Continuous-Time Version of the Principal Agent Problem" mimeo


7. Williams, N. 2003 "On Dynamic Principal-Agent Problem in continuous-time" mimeo