Computing optimal insurance contract in closed system

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Abstract

In this paper I study computational procedures of finding the set of payoff achieved in an equilibrium of a risk sharing Kocherlakota (1996) game. Instead of using some procedure involving iterations on the sets using an APS operator, I interpret the game as a renegotiable contract and then using the ergodic properties of this contract devise a computational procedure as a static optimization problem.

1 Introduction

It is a well-documented fact, that there is a big increasing income inequality in many countries. For instance, in the last 25 years in the United States there was a sharp increase in the income inequality. However, this fact alone does not provide the full picture for the study of the welfare effects. Some authors argue that because a lot of this income variation arrives from the volatile transitory component, it is hard to evaluate the lifetime resources available for individuals. This made some authors move to study the variance in the individual consumption.

For instance, the study by Krueger and Perri (2002) shows that, although the income inequality was significantly increasing in the last 25 years the variance of the personal consumption did not have that significant increase and stayed at approximately at the same level over all this period. Another empirical findings of this work is that income inequality has increased both
between and within education groups but the consumption inequality has increased between groups but has actually declined within groups. This finding suggests that the households can do better job now insuring their income risks.

However, there is no complete risk sharing. It was also pointed out by the empirical studies that, conditional on per capita consumption, individual consumption is positively correlated with lagged and current individual income. This stylized fact suggests that there should be some inefficiency in the allocation of the consumption risk. One of the possible theoretical explanations to this fact (Thomas and Warrall (1990), Atkeson and Lucas (1992), Wang (1994) etc.) is that in the presence of the costly monitoring of income and effort, this allocation might be optimal.

Another stream of literature (Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000), Krueger and Perri (2002)) offer a different explanation to the fact. They pointed out that the complete risk sharing is not achievable because individuals face individual rationality constraints and may go away not delivering up to their obligations.

Kehoe and Levine (1993) in their study theoretically characterize the debt constrained asset market where debt constraint serves as the individual rationality constraint.

Kocherlakota (1996) builds a model where agents insure each other in small groups. The argument for this model is that it may be prohibitively costly for the outside agency to learn the characteristics of the individuals, while it is quite feasible inside small groups like family, community, village etc. By observing each other and knowing each others’ characteristics the agents do not have to worry about the incentives constraints. But now the agents face the individual rationality constraints, because they cannot credibly commit to paying back in the future and always have a possibility to leave the group. One of the assumptions is that when a person quits the group his reputation is rotten and, thus, the group refuses him of smoothening his future consumption, therefore, he goes to the autarky. So, the author characterizes the optimal solution to the model and derives some of its implications.

Alvarez and Jermann (2000) in their paper study the efficiency and implications for the asset pricing of the environments introduced by the Kehoe and Levine (1993), and Kocherlakota (1996). They introduce the equilibrium concept with complete markets and endogenous solvency constraints and characterize the preferences that lead to equilibria with incomplete risk
Krueger and Perri (2002) in their work make both empirical and theoretical statements about the relationship between the income and the consumption. Empirically they document some facts about the income and consumption inequality, including those mentioned in the beginning of this introduction and then try to come up with theoretical explanations of some of the facts in the data. One of their main theoretical contributions which they claim is to show that whenever there is some sharing of idiosyncratic risks in the economy and increase in the income, keeping the persistence of the income process constant, always leads to a reduction in consumption inequality within the group that shares income risk.

In this paper I would like to show an easy way to obtain numerical solutions to the model introduced by Kocherlakota. It is quite difficult to solve it directly because it requires some fancy fixed-point estimation. However, given the characterization of the model made by Kocherlakota, and if we interpret the consumption as contracts and notice that they have a very easily tractable Markov structure, we may reformulate the problem as a static constrained optimization, which is much easier solvable using the standard numerical techniques.

In the last section I show some easy ways to use the solution of the model to illustrate some of the theoretical statements of the Krueger and Perri (2002) and Kocherlakota (1996).

2 Model characterization

The setting I consider here is very similar to the one in Kocherlakota (1996). There are two infinitely-lived agents. The endowment each of the agents gets is determined by the realization of the discrete i.i.d. random variable $\theta_t$ with support equal to \{1,2,3,...,S\}; the probability of each realization is denoted by $\pi_t$. There is a single perishable consumption good. Let us assume that the sum of the endowments of the agents is one. Therefore, the realization of the random variable determines the split of the endowment. Let $y_s$ be the endowment of the first agent, then the endowment of the second agent will be $1 - y_t$. Without the loss of generality we may assume $y_1 > y_2 > .. > y_s$.

In period $t$ the agents have the identical preferences described by the utility function:
Let us assume, unlike in Kocherlakota (1996), that both the distribution of the endowments and the endowments of the two agents are symmetric. This assumption makes the two agents identical with perfectly negatively correlated endowments. Let us, also, assume that the social planner values two agents identically. These two assumptions will largely facilitate characterization of the optimal solution.

Then, as was show in Kocherlakota (1996) the social planner’s problem can be written as:

$$P(v) = \max_{c_s,w_s} \sum_{s=1}^{S} \pi_s[U(1 - c_s) + \beta P(w_s)]$$  \hspace{1cm} (2)

$$\sum_{s=1}^{S} \pi_s[U(1 - c_s) + \beta P(w_s)] \geq v$$  \hspace{1cm} (3)

$$U(c_s) + \beta w_s \geq U(y_s) + \beta v_{aut}, s = 1,2,\ldots,S$$ \hspace{1cm} (4)

$$U(1 - c_s) + \beta P(w_s) \geq U(1 - y_s) + \beta v_{aut}, s = 1,2,\ldots,S$$ \hspace{1cm} (5)

Here the expression (3) is the promise keeping constraint with $w_s$ promised continuation utility. The two other expressions (4)(5) are individual rationality constraints for the both agents.

If we add the LaGrange multipliers $\mu, \lambda_s, \theta_s$, to the constraints (3)(4)(5) and assume differentiability of the $P(\cdot)$, the following first order conditions characterize the solution:

$$-(\pi_s + \theta_s)U'(1 - c_s) + (\lambda_s + \pi_s\mu)U'(c_s) = 0$$ \hspace{1cm} (6)

$$(\pi_s + \theta_s)P'(w_s) + (\lambda_s + \pi_s\mu) = 0$$ \hspace{1cm} (7)

and the envelope condition:

$$P'(v) = \mu$$ \hspace{1cm} (8)
These equations imply:

\[-\frac{U'(1 - c_s)}{U'(c_s)} = P'(w_s)\]  \hfill (9)

If we assume that we can achieve anything better than the autarkic level, then we may consider three possibilities: (i) constraint (4) binds, (ii) constraint (5) binds, (iii) neither of the constraints (4) (5) bind. As long as we assumed, that anything better than autarkic level is achievable, we may rule out the case when both the individual rationality constraints bind.

After some algebra we obtain the following systems of equations characterizing the optimal solutions. Possibility (i) corresponds to:

\[
\begin{align*}
U(c_s) + \beta w_s &= U(y_s) + \beta v_{aut} \\
-\frac{U'(1 - c_s)}{U'(c_s)} &= P'(w_s)
\end{align*}
\]  \hfill (10)

Possibility (ii) corresponds to the following system of equations:

\[
\begin{align*}
U(1 - c_s) + \beta P(w_s) &= U(1 - y_s) + \beta v_{aut} \\
-\frac{U'(1 - c_s)}{U'(c_s)} &= P'(w_s)
\end{align*}
\]  \hfill (11)

And if none binds then (iii):

\[
\begin{align*}
w_s &= v \\
-\frac{U'(1 - c_s)}{U'(c_s)} &= P'(w_s)
\end{align*}
\]  \hfill (12)

Note that in both cases when one of the constraints binds the new level of consumption does not depend on the past level $v$. In the third case when none of the constraints bind the consumption level stays constant. Given the past promised level $v$ there are states (i) when the ir for the first agent binds and the public planner has to increase the promised level of consumption $w_s > v$, states (ii) where the planner decreases the promised utility $w_s < v$, and states (iii) where the promised utility stays the same.

Assume that for every value $v$ the equation (9) has a unique corresponding level $c_s$ which solves the equation. This and some assumptions about $P'(\cdot)$ guarantee that if a certain ir constraint binds, there is a unique pair \{c_s, w_s\} which solves the corresponding system of equations. Consider two cases.
Suppose that we can achieve the full insurance and as we assumed earlier the social planner values both agents identically, then each agent gets 1/2 of the aggregate endowment. The necessary and sufficient condition for this case is:

\[
\frac{U(1/2)}{1 - \beta} \geq U(y_1) + \beta v_{aut}
\]  

(13)

Note, that we use the introduced sorting of the endowment: \( y_1 > y_2 > \ldots > y_s \). What this condition means is that if the individual rationality constraint does not bind in the case with the highest endowment for the equal split of endowment, then we may achieve the full insurance. Because we assumed symmetric distribution over symmetric endowments, we do not need to check the ir for the second agent.

Consider the other case when the condition (13) is not satisfied, i.e. we cannot achieve the full insurance. Let the sequence: \( \{c_{s,j}, w_{s,j}\}_{s=1}^{S} \) be the solution to the systems of equations (10), (11) where the index \( j \) shows which system the pair solves. Evidently, this sequence is finite. Assume we start off the first state, then the subsequent allocations \( \{c_t, w_t\}_{t=1}^{\infty} \) follow the Markov process with a finite support. The Markov structure of the allocations is evident, since for each \( v \) there are some states, where ir for one of the agents binds in which \( v \) becomes obsolete and we adopt a new level of promised utility \( w_s \) (independent of \( v, v \) here is used to determine the binding states) and states where none of the ir constraints bind, in which case the level of promised utility stays the same \( v \). As all the future paths depend just on \( v \) and not on any previous history, I conclude that the allocations follow the Markov process (by definition). To see that the support of the Markov process is finite we assume that we start off the 1-st state. In this case the ir for the first agent binds and then takes values off the \( \{c_{s,j}, w_{s,j}\}_{s=1}^{S} \) sequence which is finite. Because the split of the endowment is iid we are guaranteed that in the limit we shall hit either \( s = 1 \) or \( s = S \) and, thus, all the subsequent allocations will follow the outlined Markov process.

Let us consider the properties of this limiting process. It is useful now to interpret the allocations \( \{c_{s,j}, w_{s,j}\}_{s=1}^{S} \) as contracts. Assume that we start off the worst for the first agent state: \( s = S \), then we adopt some contract corresponding to the allocation: \( \{c_{S,2}, w_{S,2}\} \). In the following period we will either stay in the same allocation \( \{c_{S,2}, w_{S,2}\} \) or will transit to a new one and we know for certain that only the constraints for the 1-st agent may bind. Define \( I \) such that given the allocation \( \{c_{S,2}, w_{S,2}\} \) in the subsequent
periods for all the states below or equal to \( I \) the ir for the first agent will bind and we transit to some allocation \( \{c_{s,1}, w_{s,1}\}_{s=1}^{\infty} \), and for all the other states we stay in the same allocation. Now we see that we need \( 2I \) contracts to support the efficient allocations in this limiting process. We may go further and claim that as we assumed that the two agents are identical, we may claim that we may characterize the optimal allocation by just \( I \) contracts.

Let \( \{l_i\}_{i=1}^I \) be the \( I \) contracts corresponding to the allocations \( \{c_{s,1}, w_{s,1}\}_{s=1}^{\infty} \), such that numerically \( l_i = c_{i,1} \), then \( \{1 - l_i\}_{i=1}^I \) contracts correspond to the \( \{c_{s,2}, w_{s,2}\}_{s=S-I}^{\infty} \) allocations (Note that all analysis is done for the first agent because everything is symmetric so for the second it looks exactly the same).

Suppose there is a vector \( L \) of contracts:

\[
\begin{bmatrix}
  l_1 \\
  \vdots \\
  l_I \\
  1 - l_i \\
  \vdots \\
  1 - l_1 \\
\end{bmatrix}
\]

Then let \( \Pi \) bet the matrix defining the Markov transition rule over the optimal contracts. Clearly all the contracts in \( L \) constitute an ergotic set. Then there should be the vector with all non-negative elements \( \Pi^* \) of the invariant distribution over \( L \).

Let us show that optimal \( L \) should satisfy \( L = \arg\max \{\Pi^*U(L)\} \). Suppose the contrary, exists \( L' \neq L \) which is an optimal set of contracts and delivers strictly more utility. Then we may always break the consumption stream guaranteed by the contracts into two parts:

\[
\sum_{\tau=0}^{T} \sum_{s=1}^{S} \beta^\tau [U(c_l)] \pi_s + \sum_{\tau=T+1}^{\infty} \sum_{s=1}^{S} \beta^\tau [U(c_l)] \pi_s 
\]

Then as \( T \) increase and we converge to the ergotic distribution, the planner may always increase the utility by promising the \( L = \arg\max \{\Pi^*U(L)\} \) after \( T \) periods, and after \( T \) periods \( L \) should be the solutions to the problem (2)-(5). But we assumed that \( L' \neq L \) is optimal and delivers more utility, which is a contradiction.

Given this result it gets much easier to obtain the numeric solutions to the model. To solve the problem we have to find \( L = \arg\max \{\Pi^*U(L)\} \)
such that \( \Pi \) is consistent with the individual rationality constraints of the (2)-(5). This problem is a static constrained optimization, which is quite solvable on computer. Given the solution to the ergotic set of contracts it is straightforward to compute the optimal allocations off the ergotic set by the limit of the backward induction.

3 Computational algorithm

Given the analysis in the previous section, the problem that we have to solve is \( L = \arg \max \{ \Pi^* U(L) \} \), such that \( \Pi \) is consistent with the individual rationality constraints of the (2)-(5) problem.

Then, the computational strategy would be, first to guess \( I \), given \( I \) we may guess the Markovian transition matrix: \( \Pi \). To do this we need to use some further analysis of the model. Suppose, that the agents signed the contract \( l_i \), then in the next state we will either stay in the same contract or will transition to a new one. The number of contracts to which we can transition in the next period is not the all set \( L \), but some subset of it. We may rule out right away all the contracts belonging to the following set: \( \{l_j\}_{j=i+1}^I \). It is quite obvious to see that this is true, if we notice, that in order to transition to a new contract in the set \( \{l_j\}_{j=i+1}^I \), the corresponding individual rationality constraint for the first agent should bind. But all the individual rationality constraints in this set are automatically satisfied. By the same reasoning all the individual rationality constraints corresponding to the contracts in the set of \( \{l_j\}_{j=1}^{i-1} \) should bind.

We may also transition to the contracts where the ir for the second agent binds \( \{1 - l_i\}_{i=1}^I \). Here there should also be some consistency satisfied. Suppose that given the contract \( l_i \) we may transition to the set \( \{1 - l_i\}_{i=1}^k \) which implies that all the individual rationality constraints for the second agent corresponding to the set of contracts \( \{1 - l_j\}_{j=k+1}^I \) do not bind. Then, for all the contracts \( l_m (s.t. m > i) \) the contracts in the set of \( \{1 - l_j\}_{j=k+1}^I \) are not achievable either. And for all the contracts \( l_m (s.t. m < i) \) the contracts in the set of \( \{1 - l_i\}_{i=1}^k \).

Consider the following example. Suppose, \( S = 4, I = 3 \), the \( L' \) is:

\[
\begin{bmatrix}
l_1 & l_2 & l_3 & \text{n.a.} & \text{n.a.} & 1 - l_3 & 1 - l_2 & 1 - l_1 \\
\end{bmatrix}
\]  

(15)

one of the possible transition matrices consistent with the analysis above might be:
Suppose that we had the contract \( l_2 \), then according to this transition matrix, if in the next period \( s = 1 \) we transit to \( l_1 \), if \( s = 2 \) or \( s = 3 \) we stay at \( l_2 \), and if \( s = 4 \) we transit to \( 1 - l_1 \).

Given the guess for the transition matrix we may choose \( L = \text{arg max} \{ \Pi^*U(L) - p(L, \Pi) \} \), where \( p(L, \Pi) \) is the punishment for the inconsistencies with the individual rationality scheme implied by the \( \{L, \Pi\} \) pair. That is \( \Pi \) implies that certain constraints bind, certain don’t bind and certain hold equal. If it is inconsistent, then we multiply the difference by a big number and subtract form the objective function. By going through all possible \( I \)’s and \( \Pi \)’s we obtain the optimal solution.

### 4 Simulation results

Using the outlined algorithm we may run the simulation of the model. Let there be log preferences. Let us set \( S=4 \) and the split of endowment be: 
\[
\begin{bmatrix}
0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 & 0.25 \\
0 & 0.5 & 0.25 & 0 & 0 & 0 & 0 & 0.25 \\
0 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0.25 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0 \\
0.25 & 0 & 0 & 0 & 0 & 0.25 & 0.5 & 0 \\
0.25 & 0.25 & 0.25 & 0 & 0 & 0.25 & 0.25 & 0.25 
\end{bmatrix}
\]

(16)

Now we may compute the optimal contracts with different values of the discount factor. On the figure1 we can see the values of the expected utilities \( \{\Pi^*U(L)\} \) (vertical axes) achieved given the optimal contract \( L \) as a function of beta (horizontal axe).

Figure 1
As was expected, as beta approaches one, the punishment for quitting gets worse and, thus, it gets easier to support the incentives constraints. On the figure 2 we can see that as beta goes down we need to employ more contracts to support the optimal insurance.

Figure 2

One of the contributions of the Kocherlakota (1996) paper was to show that in this setting the optimal insurance contract scheme guarantee that the covariance of the the lagged income and the present consumption is non negative. Now we may compute this covariance directly. On the figure 3, you may see the covariance between the income shock and the contemporaneous (line with dots), one period lagged (dots) and two period lagged consumption (simple line).
In the full insurance case, as the agents get the constant stream of consumption all the covariance terms are zero. As we move left, away from the full insurance, all the covariance terms grow, but initially we may smoothen this inequality over time. As more constraints start binding the agents lose the ability to smoothen the consumption and as they approach autarky each period’s consumption approaches to the endowment.

In this setting we may easily reproduce one of the results of the Krueger, Peri (2002), that "whenever there is some sharing of the idiosyncratic risk in the economy an increase on the volatility of income leads to a reduction in the consumption inequality within the group that shares income risk." To see this, let us set the income to be more volatile: $[(0.9, 0.1)\{0.7, 0.3\} \{0.3, 0.7\} \{0.1, 0.9\}]$. We could have done the same in a different way, i.e. leaving the consumption splits the same but, change the probability of the distribution. Note that, the expected value of the income stays the same in both cases.

In figures 4-6. you may see that now everything shifted left. The result is quite obvious, if we notice that the more volatile the income is, the lower the expected utility in the autarky is, and thus, we can more easily support the risk sharing contracts.
5 Conclusion

To conclude, I would like to point out that there is a number of ways in which the research in this area could be continued. One way would be to incorporate the model into empirical findings, calibrate the model to the actual data and consider if it delivers the right values of consumption inequality and intertemporal correlations between income and consumption.

The other way to continue the research in the area is more theoretical. It would be interesting to incorporate the hidden storage in the model and check its new properties. It is quite obvious that with the given storage the individual rationality constraints will be tighter due to several reasons. First, if there is storage the individuals will be able to smoothen their consumption in the autarchy. Second, if the agent has some savings from the previous period his reservation utility is higher, therefore the planner has to bribe him more in order for him not to go to the autarky. So, if we allow for hidden storage the agents would have incentives to save in order to gain the bargaining power in the future.
6 References


